A Formalization of Higher-Order Categories in Lean 4

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Introduction

What is a formalization project? A **formalization project** is an effort to translate mathematical definitions, theorems, and proofs into a formal language that can be verified by a computer proof assistant. In this project, we use **Lean 4**, a theorem prover and programming language developed at Microsoft Research with Leonardo de Moura as the lead developer. Lean is particularly well-suited for formalizing modern mathematics.

What is this docuemnt? This document is a **blueprint** for a formalization project in Lean 4. A blueprint serves as a roadmap for mathematical formalization, providing a human-readable mathematical exposition alongside references to the corresponding formal code. It acts as a bridge between traditional mathematical writing and computer-verified proofs, allowing us to understand the mathematical content while tracking the progress of its formalization. Refer to the official Lean Blueprint GitHub repository for more information on blueprints.

What is this project about? This project formalizes the theory of **higher-order categories** and their different presentations. Specifically, we formalize:

1. **Single-sorted categories**: Where objects, morphisms, 2-morphisms, and higher morphisms all live in a single type.
2. **Many-sorted categories**: The traditional presentation where objects, morphisms, and higher morphisms live in separate types.
3. **The equivalence between these presentations**: Proving that these two approaches of higher-order categories are equivalent.

This project is based on the work by Enric Cosme Llópez and Raul Ruiz Mora in [1].

Chapter 1

Single-sorted presentation of higher-order categories

In the single-sorted presentation of higher-order categories, objects, morphisms, 2-morphisms, and all higher morphisms live in a single set, distinguished by source and target operations at each dimension. Also, composition at each dimension is defined as a partial operation that is only defined when appropriate source-target matching conditions are satisfied. In this chapter, we present the very basic definitions and axioms of single-sorted categories and single-sorted functors between them.

Notation 1.1. Throughout the text, if not otherwise specified, we will use $\mathcal{I}$ to denote a set of indices equipped with a linear order relation $<$.

1.1 Single-sorted categories

We start by defining the necessary data for defining a single-sorted category.

Definition 1.2. Let C be a set whose elements will be called *morphisms*. The required data for defining a single-sorted $\mathcal{I}$-category structure on C consists of:

- A unary operation $\mathrm{sc}^i \colon C \to C$ for each index $i \in \mathcal{I}$, called the *source* at dimension i.
- A unary operation $\mathrm{tg}^i \colon C \to C$ for each index $i \in \mathcal{I}$, called the *target* at dimension i.
- A partial binary operation $\#^i \colon C \times C \rightharpoonup C$ for each index $i \in \mathcal{I}$, called the *composition* at dimension i.

The composition $g \#^i f$ is defined if and only if $\mathrm{sc}^i(g) = \mathrm{tg}^i(f)$, in which case we say that g and f are *composable* at dimension i.

We will represent the above data as a tuple $\mathsf{C} = (C, (\mathrm{sc}^i, \mathrm{tg}^i, \#^i)_{i \in \mathcal{I}})$.

We split the axioms of single-sorted categories into two parts: first, those concerning only one dimension at a time, and then those concerning the interaction between different dimensions. This division is made by first defining the notion of pre-single-sorted category:

Definition 1.3. A *pre-single-sorted $\mathcal{I}$-category* is a tuple $\mathsf{C} = (C, (\mathrm{sc}^i, \mathrm{tg}^i, \#^i)_{i \in \mathcal{I}})$ satisfying, for each index $i \in \mathcal{I}$, the following axioms:

- For all $f \in C$,

$$\begin{aligned} \mathrm{sc}^i(\mathrm{sc}^i(f)) &= \mathrm{sc}^i(f), \qquad & \mathrm{tg}^i(\mathrm{sc}^i(f)) &= \mathrm{sc}^i(f), \\ \mathrm{sc}^i(\mathrm{tg}^i(f)) &= \mathrm{tg}^i(f), \qquad & \mathrm{tg}^i(\mathrm{tg}^i(f)) &= \mathrm{tg}^i(f). \end{aligned}$$

- For all $f, g \in C$, if g and f are composable, then

$$\mathrm{sc}^i(g \#^i f) = \mathrm{sc}^i(f), \qquad \mathrm{tg}^i(g \#^i f) = \mathrm{tg}^i(g).$$

- For all $f \in C$, the morphisms f and $\mathrm{sc}^i(f)$ are composable and

$$f \#^i \mathrm{sc}^i(f) = f.$$

- For all $f \in C$, the morphisms $\mathrm{tg}^i(f)$ and f are composable and

$$\mathrm{tg}^i(f) \#^i f = f.$$

- For all $f, g, h \in C$, if h and g are composable, and g and f are composable, then $h \#^i g$ and f and h and $g \#^i f$ are composable, and

$$(h \#^i g) \#^i f = h \#^i (g \#^i f).$$

We finally define a single-sorted category structure by extending the above definition with the axioms concerning the interaction between different dimensions:

Definition 1.4 (Single-sorted category)**.** A *single-sorted $\mathcal{I}$-category* is a pre-single-sorted $\mathcal{I}$-category $\mathsf{C} = (C, (\mathrm{sc}^i, \mathrm{tg}^i, \#^i)_{i \in \mathcal{I}})$ satisfying, in addition to the axioms of pre-single-sorted categories, the following cross-dimensional axioms:

- For all $f \in C$ and all indices $j, k \in \mathcal{I}$ with $j < k$,

$$\begin{aligned} \mathrm{sc}^k(\mathrm{sc}^j(f)) &= \mathrm{sc}^j(f), & \mathrm{sc}^j(\mathrm{sc}^k(f)) &= \mathrm{sc}^j(f), \\ \mathrm{sc}^j(\mathrm{tg}^k(f)) &= \mathrm{sc}^j(f), & \mathrm{tg}^k(\mathrm{tg}^j(f)) &= \mathrm{tg}^j(f), \\ \mathrm{tg}^j(\mathrm{tg}^k(f)) &= \mathrm{tg}^j(f), & \mathrm{tg}^j(\mathrm{sc}^k(f)) &= \mathrm{tg}^j(f). \end{aligned}$$

- For all $f, g \in C$ and all indices $j, k \in \mathcal{I}$ with $j < k$, if g and f are composable at dimension j, then $\mathrm{sc}^k(g)$ and $\mathrm{sc}^k(f)$ and $\mathrm{tg}^k(g)$ and $\mathrm{tg}^k(f)$ are also composable at dimension j, and

$$\begin{aligned} \mathrm{sc}^k(g \#^j f) &= \mathrm{sc}^k(g) \#^j \mathrm{sc}^k(f), \\ \mathrm{tg}^k(g \#^j f) &= \mathrm{tg}^k(g) \#^j \mathrm{tg}^k(f). \end{aligned}$$

- For all $f_1, f_2, g_1, g_2 \in C$ and all indices $j, k \in \mathcal{I}$ with $j < k$, suppose that:
  - g_2 and f_2 are composable at dimension j,
  - g_1 and f_1 are composable at dimension j,
  - g_2 and g_1 are composable at dimension k, and
  - f_2 and f_1 are composable at dimension k.

  Then, the morphisms $(g_2 \#^j f_2)$ and $(g_1 \#^j f_1)$ are composable at dimension k, and the morphisms $(g_2 \#^k g_1)$ and $(f_2 \#^k f_1)$ are composable at dimension j, and

$$(g_2 \#^j f_2) \#^k (g_1 \#^j f_1) = (g_2 \#^k g_1) \#^j (f_2 \#^k f_1).$$

  That is, composing first at dimension j and then at dimension k yields the same result as composing first at dimension k and then at dimension j.

Definition 1.5 (Single-sorted finite category)**.** Given a natural number $n \in \mathbb{N}$, a *single-sorted n-category* is a single-sorted category indexed by the finite set $n = \{0, 1, \ldots, n-1\}$.

We now define the notion of cell or identity morphism at a given dimension in a single-sorted category.

Definition 1.6 (Cell)**.** Let C be a single-sorted $\mathcal{I}$-category and $k \in \mathcal{I}$. We say that a morphism $f \in C$ is a *k-cell* or *identity morphism at dimension k* if and only if

$$\mathrm{sc}^k(f) = f.$$

We will denote the set of k-cells of C by C_k.

The above definition defines k-cells using the source operation, however, it is also possible to define them using the targer operation and, using the axioms of single-sorted categories, one can prove that both definitions are equivalent.

Definition 1.7 (Target-based cell)**.** Let C be a single-sorted $\mathcal{I}$-category and $k \in \mathcal{I}$. A morphism $f \in C$ is a *target-based k-cell* if and only if

$$\mathrm{tg}^k(f) = f.$$

Theorem 1.8 (Equivalence of cell definitions)**.** *Let C be a single-sorted $\mathcal{I}$-category and $k \in \mathcal{I}$. A morphism $f \in C$ is a k-cell if and only if it is a target-based k-cell.*

It follows that the set of k-cells C_k is equal to the set of target-based k-cells.

Proof. Let $f \in C$. Suppose first that f is a k-cell, i.e., $\mathrm{sc}^k(f) = f$. By the axioms of single-sorted categories, we have that

$$\mathrm{tg}^k(f) = \mathrm{tg}^k(\mathrm{sc}^k(f)) = \mathrm{sc}^k(f) = f,$$

so f is a target-based k-cell.

Conversely, suppose that f is a target-based k-cell, i.e., $\mathrm{tg}^k(f) = f$. By the axioms of single-sorted categories, we have that

$$\mathrm{sc}^k(f) = \mathrm{sc}^k(\mathrm{tg}^k(f)) = \mathrm{tg}^k(f) = f,$$

so f is a k-cell. □

Definition 1.9 (Single-sorted omega category)**.** A *single-sorted ω-category* is a single-sorted category indexed by the set of natural numbers $\mathbb{N}$, satisfying and additional axiom: for all $f \in C$, there exists a dimension $k \in \mathbb{N}$ such that f is a k-cell.

1.2 Single-sorted functors

Unlike traditional category theory where functors act separately on objects and morphisms, here a single-sorted functor is simply a function on the set of morphisms that preserves sources, targets, and composition at each dimension.

Definition 1.10 (Single-sorted functor)**.** Let $\mathsf{C} = (C, (\mathrm{sc}^i, \mathrm{tg}^i, \#^i)_{i\in\mathcal{I}})$ and $\mathsf{D} = (D, (\mathrm{sc}^i, \mathrm{tg}^i, \#^i)_{i\in\mathcal{I}})$ be single-sorted $\mathcal{I}$-categories. A *single-sorted $\mathcal{I}$-functor* from C to D is a map $F\colon C \to D$ satisfying, for each index $k \in \mathcal{I}$, the following preservation properties:

- For all $f \in C$,
$$F(\mathrm{sc}^k(f)) = \mathrm{sc}^k(F(f)), \qquad F(\mathrm{tg}^k(f)) = \mathrm{tg}^k(F(f)).$$
- For all $f, g \in C$, if g and f are composable at dimension k, then $F(g)$ and $F(f)$ are also composable at dimension k, and
$$F(g \#^k f) = F(g) \#^k F(f).$$

We can define composition of single-sorted functors as the usual composition of functions, and the identity single-sorted functor as the identity function on the set of morphisms. The following results assert that these constructions yield valid single-sorted functors.

Theorem 1.11 (Composition of single-sorted functors)**.** *Let $\mathsf{C}, \mathsf{D}, \mathsf{E}$ be single-sorted $\mathcal{I}$-categories, and let $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{E}$ be single-sorted $\mathcal{I}$-functors. Then, the composition $G \circ F\colon \mathsf{C} \to \mathsf{E}$ is also a single-sorted $\mathcal{I}$-functor.*

Proof. Let $k \in \mathcal{I}$. For all $f \in C$, since F and G are single-sorted functors, we have that

$$(G \circ F)(\mathrm{sc}^k(f)) = G(F(\mathrm{sc}^k(f))) = G(\mathrm{sc}^k(F(f))) = \mathrm{sc}^k(G(F(f))) = \mathrm{sc}^k((G \circ F)(f)),$$

and similarly,

$$(G \circ F)(\mathrm{tg}^k(f)) = G(F(\mathrm{tg}^k(f))) = G(\mathrm{tg}^k(F(f))) = \mathrm{tg}^k(G(F(f))) = \mathrm{tg}^k((G \circ F)(f)),$$

that is, $G \circ F$ preserves sources and targets at dimension k.

Now, let $f, g \in C$ be composable at dimension k. We have that

$$\begin{aligned}(G \circ F)(g \#^k f) &= G(F(g \#^k f)) \\ &= G(F(g) \#^k F(f)) \\ &= G(F(g)) \#^k G(F(f)) \\ &= (G \circ F)(g) \#^k (G \circ F)(f),\end{aligned}$$

so $G \circ F$ preserves composition at dimension k. □

Theorem 1.12 (Identity single-sorted functor)**.** *Let* C *be a single-sorted* $\mathcal{I}$*-category. Then, the identity function* $\mathrm{id}_C \colon C \to C$ *is a single-sorted* $\mathcal{I}$*-functor from* C *to itself.*

Proof. It trivially follows from reflexivity. □

Definition 1.13 (Single-sorted finite functor)**.** Given a natural number $n \in \mathbb{N}$, a *single-sorted* n*-functor* is a single-sorted functor between single-sorted n-categories.

Definition 1.14 (Single-sorted omega functor)**.** A *single-sorted* ω*-functor* is a single-sorted functor between single-sorted ω-categories.

1.3 The category of single-sorted categories

Lastly, we define the category whose objects are single-sorted categories and whose morphisms are single-sorted functors.

Definition 1.15 (Category of single-sorted categories)**.** The *category of single-sorted* $\mathcal{I}$*-categories*, denoted by $\mathcal{I}\mathsf{Cat}$, is defined as follows:

- Its objects are single-sorted $\mathcal{I}$-categories.
- Its morphisms are single-sorted $\mathcal{I}$-functors between single-sorted $\mathcal{I}$-categories.
- Composition of morphisms is given by composition of single-sorted functors, which is well defined by Theorem 1.11.
- The identity morphism for each object is given by the identity single-sorted functor, which is well defined by Theorem 1.12.

Proof. The required category laws trivially follow from the corresponding properties of function composition. □

Notation 1.16. Since single-sorted n-categories are just single-sorted $\mathcal{I}$-categories where $\mathcal{I}$ is the finite set n, it follows that we can define the category of single-sorted n-categories and single-sorted n-functors between them, denoted by $n\mathsf{Cat}$.

Definition 1.17 (Category of single-sorted omega categories)**.** The *category of single-sorted* ω*-categories*, denoted by $\omega\mathsf{Cat}$, is defined analogously to Definition 1.15, but using single-sorted ω-categories and single-sorted ω-functors instead.

Proof. As in Definition 1.15, the required category laws trivially follow from the corresponding properties of function composition. □

Chapter 2

Underlying transformations

In this chapter, for any two naturals n and m with $m < n$, we provide a way to transform single-sorted n-categories (or ω-categories) into single-sorted m-categories by *forgetting* morphisms that are not m-cells in the original category.

Similarly, we provide a way to transform single-sorted n-functors (or ω-functors) into single-sorted m-functors by forgetting the action on morphisms that are not m-cells in the original category.

We conclude the chapter by showing that these transformations are functorial, i.e., they define a functor from $n\mathsf{Cat}$ (or $\omega\mathsf{Cat}$) to $m\mathsf{Cat}$.

2.1 Underlying single-sorted categories

Lemma 2.1. *Let C be a single-sorted $\mathcal{I}$-category and $m \in \mathcal{I}$. Then, for every $f \in \mathsf{C}$ and every $k \in \mathcal{I}$ such that $k < m$, we have that $\mathrm{sc}^k(f)$ is a m-cell in C.*

Proof. Applying one of the axioms of single-sorted categories, we have that

$$\mathrm{sc}^m(\mathrm{sc}^k f) = \mathrm{sc}^k(f).$$

□

Lemma 2.2. *Let C be a single-sorted $\mathcal{I}$-category and $m \in \mathcal{I}$. Then, for every $f \in \mathsf{C}$ and every $k \in \mathcal{I}$ such that $k < m$, we have that $\mathrm{tg}^k(f)$ is a m-cell in C.*

Proof. Applying one of the axioms of single-sorted categories, we have that

$$\mathrm{tg}^m(\mathrm{tg}^k f) = \mathrm{tg}^k(f).$$

That is, $\mathrm{tg}^k(f)$ is a target-based m-cell in C. However, we know that every target-based m-cell is also a m-cell. □

Lemma 2.3. *Let C be a single-sorted $\mathcal{I}$-category and $m \in \mathcal{I}$. Then, for every $f, g \in C_m$ (i.e., f and g are m-cells in C) and every $k \in \mathcal{I}$ such that $k < m$, if the comoposition $g \#^k f$ is defined, then $g \#^k f$ is a m-cell in C.*

Proof. Applying one of the axioms of single-sorted categories and the fact that f and g are m-cells, we have that

$$\mathrm{sc}^m(g \#^k f) = \mathrm{sc}^m(g) \#^k \mathrm{sc}^m(f) = g \#^k f.$$

□

Definition 2.4 (Underlying single-sorted category)**.** Let C be a single-sorted n-category (or ω-category) and m be a natural number such that $m < n$[1]. We define the *underlying single-sorted m-category* of C as the single-sorted m-category C_m defined on the set of m-cells of C, C_m, with the corestriction of the source, target, identity, and composition operations of C to C_m.

Note that, by Lemmas 2.1, 2.2, and 2.3, these operations are well-defined. Also, the axioms of single-sorted categories for C_m follow directly from the corresponding axioms for C.

[1] If C is an ω-category, then m can be any natural number.

2.2 Underlying single-sorted functors

Lemma 2.5. *Let* C *and* D *be single-sorted* $\mathcal{I}$*-categories,* $F\colon \mathsf{C} \to \mathsf{D}$ *be a single-sorted functor, and* $m \in \mathcal{I}$*. Then, for every* $f \in C_m$ *we have that* $F(f) \in D_m$*, i.e., the image of* m*-cells under a single-sorted functor are* m*-cells.*

Proof. Applying one of the axioms of single-sorted functors and the fact that f is a m-cell, we have that

$$\mathrm{sc}^m(F(f)) = F(\mathrm{sc}^m(f)) = F(f).$$

□

Definition 2.6 (Underlying single-sorted functor)**.** Let C and D be single-sorted $\mathcal{I}$-categories, $F\colon \mathsf{C} \to \mathsf{D}$ be a single-sorted functor, and $m \in \mathcal{I}$. We define the *underlying single-sorted* m*-functor* of F as the single-sorted m-functor $F_m\colon \mathsf{C}_m \to \mathsf{D}_m$ defined as the corestriction of F to the sets of m-cells C_m and D_m.

Note that, by Lemma 2.5, this function is well-defined. Also, the axioms of single-sorted functors for F_m follow directly from the corresponding axioms for F.

2.3 Functoriality of the underlying transformations

Bibliography

[1] Juan Climent Vidal and Enric Cosme Llópez. From higher-order rewriting systems to higher-order categorial algebras and higher-order Curry-Howard isomorphisms, 2024. Preprint. Chapter 5: Higher-order categories.